CERTAIN PROBLEMS RELATING TO THE REGULAR THERMAL REGIME OF BICOMPONENT MATERIALS

A. G. Shifel'bain

UDC 536.21

We have derived equations – convenient for practical application – describing the regular thermal regime of a plate, a sphere, and a cylinder, with a finite value for the heat-transfer coefficient when the heat capacity of the core is greater than or equal to the heat capacity of the shell.

With onset of a regular thermal regime, the temperature of a body is described by the first term in the Fourier series

$$T = U \exp(-mt), \tag{1}$$

where U is a function of the coordinates and of the initial conditions; the coefficient m or, as it is usually known, the cooling rate, is a function of the thermophysical and geometric parameters of the system.

Kondrat'ev has demonstrated that for a thermally insulated metal core the equation which relates the cooling rate with the system parameters can be derived in two ways. The first method is associated with the determination of the least of the eigenvalues for the Fourier series describing the temperature regime of the system, and it is a more rigorous method, although the equations derived by this means cannot always be simplified. The second method is based on the determination of the heat balances. The equations for the regular thermal regime of bicomponent plates and spheres when $\alpha = \infty$ were derived in [1-3].

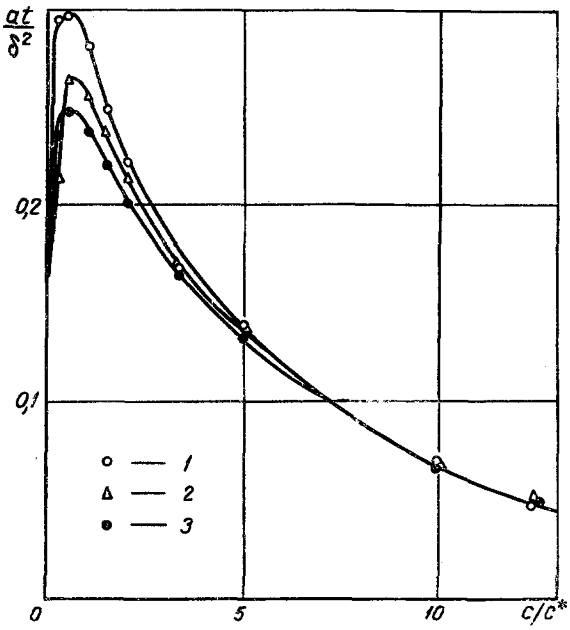


Fig. 1. The dimensionless time for the onset of the regular thermal regime as a function of the relationship between the heat capacities of the metal and the thermal insulation: 1) $T_{in} = 0$; 2) $T_{in} = T_{in}^{*}(1 - x/\delta)$; 3) $T_{in} = T_{in}^{*}$.

We consider the regular thermal regime of a plate, a sphere, a cylinder, and bodies of more complex shape, given a finite value for the heat-transfer coefficient when the heat capacity of the core is greater than or equal to the heat capacity of the shell. The equations for the plate and the sphere have been derived in two ways, which demonstrates that the method of heat balances is sufficiently accurate; the equation for the cylinder has been derived exclusively by the second method.

1. The First Method. The Symmetric Plate.

We consider the temperature field of a shell and we make provision for the effect of the metallic core only in the boundary conditions. The particular solution of the heat-conduction equation has the form

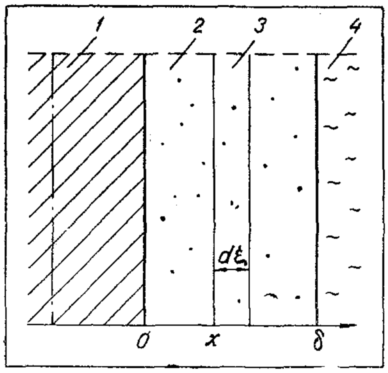
$$T = (A \cos \mu x + B \sin \mu x) \exp(-mt), \tag{2}$$

where

$$m = \mu^2 a. \tag{3}$$

The boundary conditions are the following: when x = 0

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 16, No. 2, pp. 292-298, February, 1969. Original article submitted December 11, 1967.



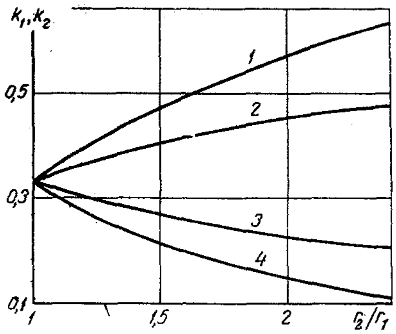


Fig. 2 Fig. 3

Fig. 2. A bicomponent plate: 1) metallic core; 2) thermal-insulation shell; 3) elemental layer of thermal insulation; 4) ambient medium.

Fig. 3. Form factors for the sphere and the cylinder as functions of the ratio between the outside and the inside radii of the shell: 1) k_s^*; 2) k_c; 3) k_c^*; 4) k_s.

there is ideal contact between the heat insulation and the metal; when $x = \delta$ the transfer of heat taking place at the outside surface follows Newton's law [4]:

$$\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{C^*}{S} \frac{\partial T^*}{\partial t}, \tag{4}$$

$$-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=\delta} = \alpha T|_{x=\delta} \tag{5}$$

(let us examine the cooling process; the temperature of the ambient medium will serve as our coordinate origin). Eliminating A and B from (2)-(5), we find the equation for the determination of the eigenvalues of μ:

$$m \frac{C^*}{S} \frac{\dfrac{\operatorname{tg} \mu\delta}{\lambda\mu} + \dfrac{1}{\alpha}}{1 - \dfrac{\lambda\mu}{\alpha} \operatorname{tg} \mu\delta} = 1. \tag{6}$$

Since the regular regime is governed by the least root of Eq. (6), the series expansion of $\tan \mu\delta$ in powers of $\mu\delta$ and retaining two terms of the expansion (we assume that $\mu\delta < 1$), after transformations we derive the equation

$$mC^* (R^\lambda + R^\alpha) + mC \left(kR^\lambda + R^\alpha - k^* \frac{R^\lambda R^\alpha}{R^\lambda + R^\alpha + \dfrac{C}{C^*} R^\alpha} \right) = 1, \tag{7}$$

where

$$R^\lambda = R_p^\lambda = \frac{\delta}{\lambda S}; \quad R^\alpha = R_p^\alpha = \frac{1}{\alpha S} \tag{8}$$

are the thermal resistances of the heat insulation and of the heat transfer at the external surface;

$$k = k_p = \frac{1}{3}; \quad k^* = k_p^* = \frac{1}{3} \tag{9}$$

are the numerical coefficients which we will refer to as form factors.

The equation for the determination of the eigenvalues of a sphere are written in the form

$$\frac{\lambda\mu^2\delta^2 r_1}{\alpha r_2} + \left(\frac{\delta}{r_2} - \frac{mC^*\delta r_1}{S\lambda r_2} \right) \left(\frac{\lambda}{\alpha r_2} - 1 \right) = \frac{\left(\dfrac{\lambda\delta}{\alpha r_2} - \dfrac{mC^* r_1}{S r_2} + \dfrac{r_1}{r_2} \right) \mu\delta}{\operatorname{tg} \mu\delta}. \tag{10}$$

After simplication, given that $\mu\delta < 1$, we find Eq. (7) in which the thermal resistances and the form factors are written as follows:

$$R^\lambda = R_s^\lambda = \frac{r_2 - r_1}{\lambda 4\pi r_1 r_2}; \quad R^\alpha = R_s^\alpha = \frac{1}{\alpha 4\pi r_2^2}, \tag{11}$$

$$k = k_s = \frac{r_1^2}{r_1^2 + r_1 r_2 + r_2^2}, \tag{12}$$

$$k^* = k_s^* = \frac{r_2^2}{r_1^2 + r_1 r_2 + r_2^2}. \tag{13}$$

We have to evaluate the area of application for Eq. (7). First of all, this equation is valid only after the onset of the regular thermal regime. It is possible to evaluate the time for the onset of the regular regime under various initial conditions from the theoretical curves of a symmetric plate when $\alpha = \infty$, as shown in Fig. 1. Moreover, in the derivation of (7) we assumed that $\mu\delta < 1$. When $\alpha = \infty$, we find that this condition corresponds to $C/C^* < 1$. For any finite value of α, the conditions $C/C^* < 1$ are even more sufficient, because the thermal process is retarded in this case, which corresponds to the least values of μ.

The accuracy of Eq. (7) for a plate can be evaluated by comparing it with the original equation (5). When $\alpha = \infty$ these equations assume the form:

$$\mu\delta \operatorname{tg} \mu\delta = 1, \tag{5*}$$

$$\frac{mC^*}{S} R^\lambda + \frac{1}{3}\frac{mC}{S} R^\lambda = 1. \tag{7*}$$

The value of $C/C^* = 1$ m calculated from (7*) differs from the m calculated from (5*) by 1.4%; with $C/C^* = 0.5$, this difference is only 0.5%.

2. The Second Method. The Symmetric Plate (Fig. 2). Let us isolate an elemental layer at a distance x from the core. The heat flow through this layer is equal to the total heat flow from the cooling core and from the thermal insulation situated between the core and the elemental layer:

$$C^* \frac{\partial T^*}{\partial t} + \int_0^x \frac{\partial T}{\partial t} c\rho S\, d\xi = \lambda S \frac{dT}{dx}. \tag{14}$$

Having divided the right- and left-hand members of (14) by λS and integrating over the entire shell thickness, we derive the equation

$$\int_0^\delta \frac{C^*}{\lambda S}\frac{\partial T^*}{\partial t} dx + \int_0^\delta \frac{dx}{\lambda S}\int_0^x \frac{\partial T}{\partial t} c\rho S\, d\xi = -(T^* - T_0). \tag{15}$$

We assume that the relationship between the shell temperature and the coordinates remains the same as in the steady-state regime. This assumption is the more valid, the smaller the heat capacity of the shell relative to the heat capacity of the core:

$$T = T^* - (T^* - T_0)\frac{\xi}{\delta}. \tag{16}$$

We determine T_0 from the equation for the heat flow at the boundary between the thermal insulation and the ambient medium:

$$C^* \frac{\partial T^*}{\partial t} + \int_0^\delta \frac{\partial T}{\partial t} c\rho S\, dx = -\alpha T_0 S. \tag{17}$$

We determine the terms containing $\partial T_0/\partial t$ by differentiating the relationship

$$\frac{T_0}{T^*} = \frac{R^\alpha}{R^\lambda + R^\alpha}, \tag{18}$$

which is a rigorous equality only in the steady-state regime or when the shell heat capacity can be neglected. After integration of (15), considering (16)-(18), and denoting

TABLE 1. Values of U(C/C*) as a Function of C/C* for Various Initial Shell Temperatures

$\frac{C}{C^*}$	U(C/C*) $T_{in}=T^*_{in}$	$T_{in}=0$	$T_{in}=T^*_{in}\left(1-\frac{x}{\delta}\right)$ †	$\frac{C}{C^*}$	U(C/C*) $T_{in}=T^*_{in}$	$T_{in}=0$	$T_{in}=T^*_{in}\left(1-\frac{x}{\delta}\right)$ *)
4	1,23	0,37	0,88	0,5	1,05	0,85	0,97
2	1,18	0,56	0,92	0,3	1,03	0,91	0,98
1,5	1,10	0,63	0,93	0,2	1,02	0,94	0,99
1	1,09	0,73	0,95	0,1	1,01	0,97	0,99
0,7	1,07	0,80	0,97	0,088	1,01	0,98	1,00

† x is the distance form the core.

$$-\frac{\partial T^*}{\partial t}\Big/T^* = m, \tag{19}$$

we derive the equation

$$mC^*(R^\lambda + R^\alpha) + mC\left(kR^\lambda + R^\alpha - k^*\frac{R^\lambda R^\alpha}{R^\lambda + R^\alpha}\right) = 1, \tag{20}$$

where R^λ, R^α, k, and k* are determined from Eqs. (8) and (9). Equation (20) differs from (7) – derived by the first method – only in terms of the second order of smallness with respect to C/C*. Similar equations are derived for a sphere and a cylinder, but for the sphere R^λ, R^α, k, and k* are determined from (11)-(13), while for the cylinder

$$R^\lambda = R^\lambda_c = \frac{\ln\frac{r_2}{r_1}}{\lambda 2\pi z};\quad R^\alpha = R^\alpha_c = \frac{1}{\alpha 2\pi r_2 z}, \tag{21}$$

$$k = k_c = \frac{1}{2\ln^2\frac{r_2}{r_1}} - \frac{r_1^2}{(r_2^2 - r_1^2)\ln\frac{r_2}{r_1}} - \frac{r_1^2}{r_2^2 - r_1^2}, \tag{22}$$

$$k^* = k^*_c = \frac{1}{2\ln^2\frac{r_2}{r_1}} - \frac{r_2^2}{(r_2^2 - r_1^2)\ln\frac{r_2}{r_1}} + \frac{r_2^2}{r_2^2 - r_1^2}. \tag{23}$$

The constants k and k* for the sphere and the cylinder are shown graphically in Fig. 3. With an increase in the sphere and cylinder radii for a constant shell thickness the sphere and cylinder equations change into the equation for the plate.

3. The Regular Thermal Regime for Materials of Complex Shape. In certain cases the equation for the regular thermal regime of materials of complex shape, made up of a core in a thermal-insulation shell, can be set up on the basis of the equations applicable to the simplest of configurations. Let us consider the special case of a cylinder which terminates in hemispheres. Let us divide this cylinder into three parts: the upper hemisphere, the cylinder, and the lower hemisphere; these can be regarded as adiabatically isolated parts of bicomponent spheres and an unbounded cylinder (within the metal the location of the boundary of separation is quite arbitrary). Let us write the equation of the regular thermal regime for each of the parts, bearing in mind that the corresponding form factors remain without change, while the thermal resistances of the hemispheres are half those of the corresponding spheres:

$$\begin{aligned} m_1C_1^*R_1 + m_1C_1R_1^* &= 1,\\ m_2C_2^*R_2 + m_2C_2R_2^* &= 1,\\ m_3C_3^*R_3 + m_3C_3R_3^* &= 1. \end{aligned} \tag{24}$$

Retaining $m_1C_1^*$, $m_2C_2^*$, and $m_3C_3^*$, respectively, in the first terms, summing these equations, and bearing in mind that $m_1 = m_2 = m_3 = m$ and $C_1^* + C_2^* + C_3^* = C^*$, we obtain

$$mC^* + mC_1\frac{R_1^*}{R_1} + mC_2\frac{R_2^*}{R_2} + mC_3\frac{R_3^*}{R_3} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}. \tag{25}$$

If the upper and the lower hemispheres are identical, Eq. (25) is somewhat simplified. In analogous fashion

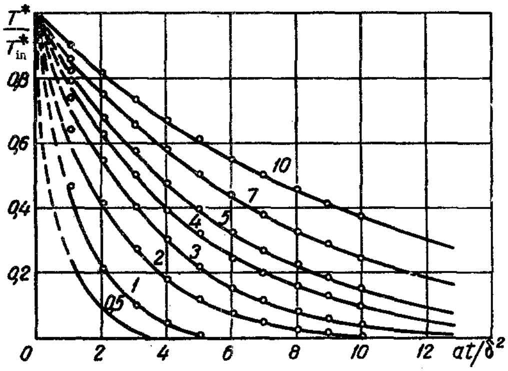


Fig. 4. Comparison of the working data for the determination of the temperature of the thermally insulated metal reservoirs as a function of time. The solid lines show the result of a rigorous solution [5], and the points have been plotted according to calculations with (27). The numerals at the curves give the values of C*/C.

we can compile the equation for the regular thermal regime of a nonsymmetric plate, as derived by Kondrat'ev in a more complex procedure. The comparative simplicity of these cases is explained by the fact that these configurations are easily divided into their component parts of the simplest shape. In other cases such as, for example, a cylinder with flat bottoms, it becomes necessary to take the edge effect into consideration. The correction factor for the edge effect can be introduced into the corresponding thermal resistance.

4. Cooling (Heating) Estimates for Bicomponent Materials, Based on the Equations for the Regular Thermal Regime. In actual practice, we frequently find it necessary to calculate the temperature of the metallic heat-insulation core in connection with processes of spontaneous cooling or heating. Since we have assumed that the core temperature is independent of the coordinates, following the onset of the regular thermal regime, the core temperature is given by

$$T^* = T^*_{in} U\left(\frac{C}{C^*}\right) \exp(-mt), \tag{26}$$

whose right-hand member is the first term of the Fourier series describing the temperature regime of the system.

We know the cooling of a core when the heat capacity of the shell is small in comparison of the heat capacity of the core follows the law

$$T^* = T^*_{in} \exp(-mt), \tag{27}$$

where m is given from the equations for the regular thermal regime.

Is it impossible to use (27) to estimate the core temperature even in those cases in which the heat capacity of the shell is comparable to that of the core? It develops that the accuracy of the calculations depends in great measure on the initial condition; and since we assume that the initial core temperature in all cases is equal to T^*_{in}, and that the temperature of the ambient medium is equal to zero, the initial conditions are represented by the temperature of the insulation material prior to the onset of cooling. Table 1 shows the values for $U(C/C^*)$ for a plate when $\alpha = \infty$ for three different initial conditions.

It follows from Table 1 that for the purpose of tentative temperature calculations the most favorable case is the one in which the temperature distribution in the insulation is kept steady prior to the onset of cooling. This is also confirmed in Fig. 4 which shows a comparison of the calculation results for the heating of reservoirs for cryogenic liquids, these calculations based on the rigorous solution of (4) and (27). However, of greatest interest are the cases for which mathematical solution of the problem is difficult.

NOTATION

m	is the cooling rate;
t	is the time;
T^*	is the core temperature;
T^*_{in}	is the core temperature prior to the onset of cooling;
T	is the shell temperature;
T_{in}	is the shell temperature prior to the onset of cooling;
T_0	is the shell temperature at the boundary with the ambient medium;
C^*	is the heat capacity of the core;
C	is the heat capacity of the shell;
δ	is the shell thickness;
r_1, r_2	are, respectively, the inside and outside radii of the shell;
c, ρ, λ, a	are the characteristics of the thermal insulation;
α	is the heat-transfer coefficient;
S	is the area separating the metal and the thermal insulation;

R^{λ}, R^{α} are, respectively, the thermal resistances of the thermal insulation and of the heat transfer;

$R = R^{\lambda} + R^{\alpha}$;

$R^* = kR^{\lambda} + R^{\alpha} - R^{\lambda}R^{\alpha}/(R^{\lambda} + R^{\alpha})$;

k, k* are the form factors.

Subscripts

p denotes the plate;

s denotes the sphere;

c denotes the cylinder.

LITERATURE CITED

1. G. M. Kondrat'ev, The Regular Thermal Regime [in Russian], GITTL (1956).
2. A. F. Begunkova, Inzh.-Fiz. Zh., 3, No. 9 (1960).
3. A. G. Shifel'bain, Inzh.-Fiz. Zh., 11, No. 2 (1966).
4. G. Carslaw and J. Jaeger, The Conduction of Heat in Solids [Russian translation], Nauka (1964).
5. F. Kreith, J. W. Dean, and L. Brooks, Adv. Cryog. Eng., 8, 536-543 (1963).